Available online at www.sciencedirect.com

ScienceDirect

J. Math. Anal. Appl. 334 (2007) 1386–1399

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

On upper and lower bounds of higher order derivatives for solutions to the 2D micropolar fluid equations

Bo-Qing Dong [a,*], Zhi-Min Chen [b]

[a] *School of Mathematical Sciences, Nankai University, Tianjin 300071, PR China*

[b] *School of Engineering Sciences, Ship Science, University of Southampton, Southampton SO17 1BJ, United Kingdom*

Received 20 May 2006

Available online 25 January 2007

Submitted by M.C. Nucci

Abstract

The present paper is concerned with asymptotic behaviours of the solutions to the micropolar fluid motion equations in $\mathbf{R}^2$. Upper and lower bounds are derived for the L^2 decay rates of higher order derivatives of solutions to the micropolar fluid flows. The findings are mainly based on the basic estimates of the linearized micropolar fluid motion equations and generalized Gronwall type argument.

Keywords: Micropolar fluid motion equations; Upper and lower bounds; L^2 decay

1. Introduction

Consider the two-dimensional mathematical model of the incompressible viscous flows governed by the micropolar fluid equations

$$\left.\begin{aligned}&\nabla\cdot v=0,\\&\frac{\partial}{\partial t}v-(\nu+\kappa)\Delta v-2\kappa\nabla\times w+\nabla\pi+v\cdot\nabla v=0,\\&\frac{\partial}{\partial t}w-\gamma\Delta w+4\kappa w-2\kappa\nabla\times v+v\cdot\nabla w=0\end{aligned}\right\}\tag{1.1}$$

* Corresponding author.
E-mail address: bqdong@mail.nankai.edu.cn (B.-Q. Dong).

0022-247X/$ – see front matter
doi:10.1016/j.jmaa.2007.01.047

associated with the initial conditions

$$v|_{t=0} = v_0, \qquad w|_{t=0} = w_0. \tag{1.2}$$

Here $v = (v_1, v_2)$, π and w denote the unknown velocity vector field, the scalar pressure field and the scalar gyration field over the fluid-time domain $\mathbf{R}^2 \times (0, \infty)$. $\nu > 0$ is a kinematic viscosity coefficient, $\kappa \geqslant 0$ and $\gamma > 0$ are gyration viscosity coefficients, and

$$\nabla \times v = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}, \qquad \nabla \times w = \left(\frac{\partial w}{\partial x_2}, -\frac{\partial w}{\partial x_1}\right).$$

This micropolar fluid motion model, a special model of microfluid motions, was introduced by Eringen [10] and exhibits micro-rotational effects and micro-rotational inertia. Physically it may represent adequately the fluids consisting of bar-like elements. Certain anisotropic fluids, e.g. liquid crystals made up of dumbbell molecules, are of the type. When the scalar gyration vector field is neglected, the micropolar fluid motion model reduces to the incompressible Navier–Stokes equations (refer to [16,30]).

There is a large literature on the mathematical theory of micropolar fluid equations. The existence and uniqueness problems were extensively studied by Lange [17], Galdi and Rionero [11], Sava [25], and recently, by Łukaszewicz [19], Resndiz and Rojas-Medar [24] and references therein. As for the dynamical behavior of (1.1)–(1.2), one may also refer to the study by Chen and Dong [5,6], Łukaszewicz [20,21].

The time decay rate estimates of the Navier–Stokes equations were originally from the celebrated work of Leray [18]. Algebraic time decay properties of weak solutions were successfully derived by Schonbek [26–28], Kajikiya and Miyakawa [14] and Wiegner [31]. Especially, by applying a Fourier splitting method, Schonbek [26] obtained the following upper–lower bound result

$$c(1+t)^{-\frac{n}{4}} \leqslant \|u(t)\| \leqslant c_1(1+t)^{-\frac{n}{4}} \quad \text{if} \int u_0\,dx \neq 0.$$

By assuming some other nonzero condition on the initial velocity field u_0, she [27] improved the decay rates described as $(1+t)^{-\frac{n+2}{4}}$. Furthermore, by a combination of the Fourier splitting method and L^q decay estimates (see Kato [15]), Schonbek [28] also established the decay estimates of higher order derivatives of solutions to the two-dimensional Navier–Stokes equations expressed as

$$\|\nabla^m u(t)\| \leqslant c(1+t)^{-\frac{m+1}{2}},$$

where ∇^m represents any differential operator in the form $(\partial_{x_1})^{\alpha_1}(\partial_{x_2})^{\alpha_2}$ for $\alpha_1 + \alpha_2 = m$. Moreover, based on so-called Gevrey estimates, Oliver and Titi [23] studied the upper and lower bounds of higher order derivatives of Navier–Stokes equations in $\mathbf{R}^n$ with the aid of the L^2 decay rate estimates

$$c_5(1+t)^{-\frac{m+\gamma_0}{2}} \leqslant \|\nabla^m u(t)\| \leqslant c_6(1+t)^{-\frac{m+\gamma_0}{2}},$$

where γ_0 is the decay rate of the linear heat equation solution.

The interested readers may also refer to [1–4,12,13,22] for other interesting time decay problems of Navier–Stokes equations and [7–9] for time decay problems of non-Newtonian flows.

However, not much work on the time decay problem of the two-dimensional micropolar fluid motion equations is published. Moreover, compared with the Navier–Stokes equations, an extra difficulty arises in the examination of the micropolar fluid motion model due to the occurrence of the gyration vector field.

The objective of this paper is to show the upper–lower bound estimates for the L^2 decay rates of higher order derivatives of solutions to the micropolar fluid motion equations. To deduce the estimates, we examine the decay estimates of derivatives for the solutions to the linearized micropolar fluid motion equations and then extend the estimates on linearized equations to the nonlinear equations by using a generalized Gronwall type argument.

To simplify the form of (1.1), we denote

$$u = \begin{pmatrix} v_1 \\ v_2 \\ w \end{pmatrix}, \qquad u_0 = \begin{pmatrix} v_0 \\ w_0 \end{pmatrix} = \begin{pmatrix} v_{10} \\ v_{20} \\ w_0 \end{pmatrix}, \qquad D = \begin{pmatrix} \partial_1 \\ \partial_2 \\ 0 \end{pmatrix},$$

and apply formally the operator $\nabla\Delta^{-1}\nabla\cdot$ to the second equation of (1.1) to obtain the expression of the pressure force

$$\nabla\pi = -\nabla\Delta^{-1}\nabla\cdot(v\cdot\nabla v).$$

We thus introduce a new operator in the form

$$Pu = u - D\Delta^{-1}D\cdot u,$$

which is a bounded projection mapping from the Lebesgue space $L^r(\mathbf{R}^2)$ onto the space

$$L^r_\sigma\big(\mathbf{R}^2\big) = \big\{u \in L^r\big(\mathbf{R}^2\big);\ D\cdot u = 0 \text{ in the sense of distributions}\big\}, \quad 1 < r < \infty,$$

due to L^r-estimate with respect to the Laplace operator Δ.

With the use of these expressions, we formulate (1.1) in the following dynamical system

$$\frac{\partial}{\partial t}u + Au + P(v\cdot\nabla u) = 0, \tag{1.3}$$

where A is the self-conjugate and linear differential operator

$$A = \begin{pmatrix} -(\nu+\frac{\kappa}{2})\Delta & 0 & -\kappa\partial_2 \\ 0 & -(\nu+\frac{\kappa}{2})\Delta & \kappa\partial_1 \\ \kappa\partial_2 & -\kappa\partial_1 & (2\kappa-\gamma\Delta) \end{pmatrix}.$$

The linearized micropolar flow is now governed by the system

$$\frac{\partial}{\partial t}u + Au = 0, \tag{1.4}$$

and is expressed in the form as a semigroup $e^{-tA}u_0$ by taking into account the following initial value condition:

$$u(0,x) = u_0(x). \tag{1.5}$$

With the use of the notation described in Section 2, we can now summarize the main result of this paper as follows.

Theorem 1.1. *Suppose that $u(t)$ is a solution of micropolar fluid motion equations* (1.1)–(1.2) *with initial vector field $u_0 \in H^m$ $(m \geqslant 0)$ satisfying*

$$\rho(r) \equiv \int_0^{2\pi} \big|\hat{u}_0(r\theta)\big|\,d\theta = cr^{2\gamma-2} + o\big(r^{2\gamma-2}\big) \quad \textit{as } r \to 0, \tag{1.6}$$

for a constant $1 < \gamma < 2$, *then there exist two positive constants* c *and* c_1 *such that*

$$c(1+t)^{-\frac{m+\gamma}{2}} \leqslant \left\|\nabla^m u(t)\right\|_2 \leqslant c_1(1+t)^{-\frac{m+\gamma}{2}}, \quad \textit{for large } t. \tag{1.7}$$

It should be mentioned that the upper and lower bounds of solutions here are optimal in the sense that they coincide with the decay rates of the heat flow $e^{\Delta t}u_0$.

This paper is organized as follows. In Section 2, we investigate the basic estimates for the linearized micropolar fluid motion equations. In Section 3, we present the auxiliary decay estimates of micropolar fluid motion equations. Section 4 is devoted to the proof of upper bound described Theorem 1.1, whereas Section 5 is contributed to the derivation of the lower bound expressed in Theorem 1.1.

2. Estimates on the linearized equations

We use the following usual notation:

$\|\cdot\|_p =$ the norm of the Lebesgue space $L^p(\mathbf{R}^2)$,

$\|\cdot\| =$ the norm $\|\cdot\|_2$,

$H^k =$ the Hilbert space $\{u \in L^2(\mathbf{R}^2);\ \|\nabla^k u\| < \infty\}$.

The Fourier transformation of a function f is denoted by

$$Fu(\xi) = \hat{u}(\xi) = \int\limits_{R^2} e^{-\mathrm{i}x\cdot\xi} u\,dx, \quad \mathrm{i} = \sqrt{-1},$$

and $c, c_1, c_2, \ldots,$ are generic positive constants which may only depend on the initial data u_0 and may vary from line to line.

Letting the matrix

$$\Lambda = \begin{pmatrix} (\nu+\frac{\kappa}{2})|\xi|^2 & 0 & -\kappa \mathrm{i}\xi_2 \\ 0 & (\nu+\frac{\kappa}{2})|\xi|^2 & \kappa\mathrm{i}\xi_1 \\ \kappa\mathrm{i}\xi_2 & -\kappa\mathrm{i}\xi_1 & (2\kappa+\gamma|\xi|^2) \end{pmatrix},$$

and applying the Fourier transform to (1.4)–(1.5), we have

$$\frac{\partial}{\partial t}\hat{u} + \Lambda\hat{u} = 0, \qquad \hat{u}(0) = \hat{u}_0, \tag{2.1}$$

or the semigroup

$$Fe^{-At}u_0 = e^{-\Lambda t}\hat{u}_0.$$

We now present upper–lower bound estimates with respect to the semigroup.

Lemma 2.1. *Let* $\sigma_1 = 1/2\min\{\nu,\gamma\}$, $\sigma_2 = 2(\max\{\nu,\gamma\}+\kappa/2)$. *Then the estimates*

$$e^{-\sigma_2|\xi|^2 t}|\hat{u}_0| \leqslant \left|e^{-\Lambda t}\hat{u}_0\right| \leqslant e^{-\sigma_1|\xi|^2 t}|\hat{u}_0|, \tag{2.2}$$

$$\left\|\nabla^m e^{-At}u_0\right\| \leqslant ct^{-\frac{m}{2}-\frac{1}{2}}\|u_0\|_1, \tag{2.3}$$

$$\left\|\nabla e^{-At}u_0\right\| \leqslant ct^{-\frac{3}{4}}\|u_0\|_{\frac{4}{3}} \tag{2.4}$$

hold true.

Proof. Let us begin with the derivation of the eigenvalues of the matrix Λ or the roots of the following spectral polynomial:

$$
\begin{aligned}
0 &= \begin{vmatrix} (\nu+\frac{\kappa}{2})|\xi|^2-\lambda & 0 & -\kappa \mathrm{i}\xi_2 \\ 0 & (\nu+\frac{\kappa}{2})|\xi|^2-\lambda & \kappa \mathrm{i}\xi_1 \\ \kappa \mathrm{i}\xi_2 & -\kappa \mathrm{i}\xi_1 & (2\kappa+\gamma|\xi|^2)-\lambda \end{vmatrix} \\
&= \left(\left(\nu+\frac{\kappa}{2}\right)|\xi|^2-\lambda\right)\left(\left(\left(\nu+\frac{\kappa}{2}\right)|\xi|^2-\lambda\right)\left(2\kappa+\gamma|\xi|^2-\lambda\right)-\kappa^2\xi_1^2\right) \\
&\quad -\kappa^2\xi_2^2\left(\left(\nu+\frac{\kappa}{2}\right)|\xi|^2-\lambda\right) \\
&= \left(\left(\nu+\frac{\kappa}{2}\right)|\xi|^2-\lambda\right)\left(\left(\left(\nu+\frac{\kappa}{2}\right)|\xi|^2-\lambda\right)\left(2\kappa+\gamma|\xi|^2-\lambda\right)-\kappa^2|\xi|^2\right) \\
&= \left(\left(\nu+\frac{\kappa}{2}\right)|\xi|^2-\lambda\right) \\
&\quad \times\left\{\lambda^2-\left(\left(2\kappa+\gamma|\xi|^2\right)+\left(\nu+\frac{\kappa}{2}\right)|\xi|^2\right)\lambda+2\nu\kappa|\xi|^2+\gamma\left(\nu+\frac{\kappa}{2}\right)|\xi|^4\right\}.
\end{aligned}
$$

We thus have the three different eigenvalues

$$
\begin{aligned}
\lambda_1 &= \left(\nu+\frac{\kappa}{2}\right)|\xi|^2, \\
\lambda_2 &= \frac{1}{2}\left(\left(2\kappa+\gamma|\xi|^2\right)+\left(\nu+\frac{\kappa}{2}\right)|\xi|^2\right) \\
&\quad -\frac{1}{2}\sqrt{\left(\left(2\kappa+\gamma|\xi|^2\right)+\left(\nu+\frac{\kappa}{2}\right)|\xi|^2\right)^2-4\left(2\nu\kappa|\xi|^2+\gamma\left(\nu+\frac{\kappa}{2}\right)|\xi|^4\right)} \\
&= \frac{1}{2}\left(\left(2\kappa+\gamma|\xi|^2\right)+\left(\nu+\frac{\kappa}{2}\right)|\xi|^2\right) \\
&\quad -\frac{1}{2}\sqrt{\left(\left(2\kappa+\gamma|\xi|^2\right)-\left(\nu+\frac{\kappa}{2}\right)|\xi|^2\right)^2+4\kappa^2|\xi|^2}, \\
\lambda_3 &= \frac{1}{2}\left(\left(2\kappa+\gamma|\xi|^2\right)+\left(\nu+\frac{\kappa}{2}\right)|\xi|^2\right) \\
&\quad +\frac{1}{2}\sqrt{\left(\left(2\kappa+\gamma|\xi|^2\right)-\left(\nu+\frac{\kappa}{2}\right)|\xi|^2\right)^2+4\kappa^2|\xi|^2},
\end{aligned}
$$

and their corresponding eigenvectors

$$
\begin{aligned}
\tilde{\mathbf{e}}_1 &= (\xi_1,\xi_2,0), \\
\tilde{\mathbf{e}}_2 &= \left(\kappa \mathrm{i}\xi_2, -\kappa \mathrm{i}\xi_1, \left(\nu+\frac{\kappa}{2}\right)|\xi|^2-\lambda_2\right), \\
\tilde{\mathbf{e}}_3 &= \left(\kappa \mathrm{i}\xi_2, -\kappa \mathrm{i}\xi_1, \left(\nu+\frac{\kappa}{2}\right)|\xi|^2-\lambda_3\right).
\end{aligned}
$$

Setting

$$\mathbf{e}_l = \tilde{\mathbf{e}}_l/|\tilde{\mathbf{e}}_l|, \quad \text{for } l = 1, 2, 3,$$

and

$$E = \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \\ \mathbf{e}_3 \end{pmatrix}^{\mathrm{T}}$$

with $^{\mathrm{T}}$ the transpose transform, we have the spectral decomposition

$$\Lambda E = EG, \quad \text{for } G = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

and the isometric property

$$E\bar{E}^T = I \quad \text{or} \quad |Eu| = |u|,$$

since A is an self-conjugate operator.

Letting $\sigma_1 = 1/2\min\{\nu,\gamma\}$, $\sigma_2 = 2(\max\{\nu,\gamma\} + \kappa/2)$, we have the crucial relations

$$\sigma_1|\xi|^2 < \lambda_l < \sigma_2|\xi|^2, \quad \text{for } l = 1, 2, 3.$$

Therefore, we have

$$\left|e^{-At}\hat{u}_0\right| = \left|Ee^{-Gt}\bar{E}^T\hat{u}_0\right| = \left|e^{-Gt}\bar{E}^T\hat{u}_0\right|$$

and

$$e^{-\sigma_2|\xi|^2t}|\hat{u}_0| = e^{-\sigma_2|\xi|^2t}\left|\bar{E}^T\hat{u}_0\right| \leqslant \left|e^{-Gt}\bar{E}^T\hat{u}_0\right| \leqslant e^{-\sigma_1|\xi|^2t}\left|\bar{E}^T\hat{u}_0\right| = e^{-\sigma_1|\xi|^2t}|\hat{u}_0|,$$

and thus (2.2) is derived.

Furthermore, by applying Plancherel's theorem and (2.2), we have

$$\begin{aligned}
\left\|\nabla^m e^{-At}u_0\right\|^2 = \left\|\widehat{\nabla^m e^{-At}}u_0\right\|^2 &= \int_{\mathbf{R}^2} |\xi|^{2m}\left|\widehat{e^{-At}u_0}\right|^2 d\xi \\
&\leqslant c\int_{\mathbf{R}^2} |\xi|^{2m}e^{-2\sigma_1|\xi|^2t}|\hat{u}_0|^2\,d\xi \leqslant c\|u_0\|_1^2\int_{\mathbf{R}^2}|\xi|^{2m}e^{-2\sigma_1|\xi|^2t}\,d\xi \\
&\leqslant ct^{-m-1}\|u_0\|_1^2\int_0^\infty s^m e^{-s}\,ds = ct^{-m-1}\|u_0\|_1^2.
\end{aligned}$$

This implies (2.3). Similarly, we have

$$\left\|\nabla e^{-At}u_0\right\| \leqslant ct^{-\frac{1}{2}}\|u_0\|.$$

In order to show the validity of (2.4), we use the self-conjugate property of the operator A to obtain, for two vector fields u_0 and u_0',

$$\begin{aligned}
\left|\int_{\mathbf{R}^2}\left(e^{-tA}u_0\right)\cdot u_0'\,dx\right| &= \left|\int_{\mathbf{R}^2} u_0\cdot\left(e^{-tA}u_0'\right)dx\right| \\
&\leqslant \|u_0\|_{\frac{4}{3}}\left\|e^{-tA}u_0'\right\|_4 \leqslant \|u_0\|_{\frac{4}{3}}\left\|e^{-tA}u_0'\right\|^{\frac{1}{2}}\left\|\nabla e^{-tA}u_0'\right\|_2^{\frac{1}{2}} \\
&\leqslant t^{-\frac{1}{4}}\|u_0\|_{\frac{4}{3}}\left\|u_0'\right\|.
\end{aligned}$$

This implies

$$\left\|e^{-tA}u_0\right\| \leqslant ct^{-\frac{1}{4}}\|u_0\|_{L_{\frac{4}{3}}}$$

and so the validity of (2.4) due to (2.3) and the semigroup property of e^{-tA}.

The proof of Lemma 2.1 is complete. □

With the estimates of semigroup desired in Lemma 2.1, we now begin to investigate the upper and lower bounds of decay of rates to the linearized equations (1.4)–(1.5).

Lemma 2.2. *Suppose that $e^{-At}u_0$ is a solution of* (1.4)–(1.5) *with initial vector field $u_0 \in H^m$ ($m \geqslant 0$) satisfying* (1.6) *for $\gamma > 1$. Then there exist two positive constants c and c_1 such that*

$$c(1+t)^{-\frac{m+\gamma}{2}} \leqslant \left\|\nabla^m e^{-At}u_0\right\| \leqslant c_1(1+t)^{-\frac{m+\gamma}{2}}, \quad \textit{for large } t. \tag{2.5}$$

Proof. The main idea of the proof is borrowed from Oliver and Titi [23]. Applying Plancherel's theorem and (1.6), (2.2), one shows that

$$\begin{aligned}
\left\|\nabla^m e^{-At}u_0\right\|^2 &= \left\|\widehat{\nabla^m e^{-At}}u_0\right\|^2 = \int_{\mathbf{R}^2} |\xi|^{2m}\left|\widehat{e^{-At}u_0}\right|^2 d\xi \\
&\leqslant c\int_{\mathbf{R}^2} |\xi|^{2m} e^{-2\sigma_1|\xi|^2 t}\left|\hat{u}_0(\xi)\right|^2 d\xi \\
&\leqslant c\int_0^{\infty}\int_0^{2\pi} r^{2m+1}e^{-2\sigma_1 r^2 t}\left|\hat{u}_0(r\theta)\right|^2 d\theta\, dr \\
&\leqslant c\int_0^{t^{-\frac{1}{4}}} r^{2m+1}e^{-2\sigma_1 r^2 t}\rho(r)\, dr + ce^{-2\sigma_1 t^{\frac{1}{2}}}\int_{t^{-\frac{1}{4}}}^{\infty}\int_0^{2\pi} r^{2m+1}\left|\hat{u}_0(r\theta)\right|^2 d\theta\, dr \\
&\leqslant ct^{-m-\gamma}\int_0^{2\sigma_1 t^{-\frac{1}{2}}} s^{m+\gamma-1}e^{-s}\left[1+\left(\frac{o(s/t)}{s/t}\right)^{\gamma-1}\right] ds + ce^{-2\sigma_1 t^{\frac{1}{2}}}\left\|\nabla^m u_0\right\|^2,
\end{aligned}$$

for large t. The function in between the square brackets is bounded and independent of t. This implies the upper bounds in (2.5). The lower bounds are derived from the following derivation:

$$\begin{aligned}
\left\|\nabla^m e^{-At}u_0\right\|^2 &= \left\|\widehat{\nabla^m e^{-At}}u_0\right\|^2 = \int_{\mathbf{R}^2} |\xi|^{2m}\left|\widehat{e^{-At}u_0}\right|^2 d\xi \\
&\geqslant c\int_{\mathbf{R}^2} |\xi|^{2m}\left|e^{-2\sigma_2|\xi|^2 t}\left|\hat{u}_0(\xi)\right|\right|^2 d\xi \\
&= c\int_0^{\infty} r^{2m+1}e^{-2\sigma_2 r^2 t}\rho(r)\, dr
\end{aligned}$$

$$\geqslant ct^{-m-\gamma}\int_0^1 s^{m+\gamma-1}e^{-s}\,ds + o\big(t^{-m-\gamma}\big) \geqslant ct^{-m-\gamma}, \quad \text{for large } t.$$

Thus the proof of Lemma 2.2 is complete. □

3. Auxiliary decay estimates

In this section we present some auxiliary decay estimates that will be used in the following sections.

Lemma 3.1. *Under the same conditions stated in Theorem* 1.1*, we have the following auxiliary decay estimates*:

$$t\big\|\nabla u(t)\big\|^2 \to 0, \quad t\to\infty, \tag{3.1}$$

$$\big\|u(t)\big\| \leqslant c(1+t)^{-\mu}, \quad 0<\mu<1-\gamma/2, \textit{ for large } t. \tag{3.2}$$

Proof. It should be mentioned that the smooth solutions of the two-dimensional micropolar fluid equations (1.1)–(1.2) are globally defined under the smooth initial data [11,19,24]. Therefore we take L^2-inner product of (1.1) with the vector fields (v, w) and integrate by parts to get

$$\begin{aligned}
\frac{d}{dt}\big\|u(t)\big\|^2 &\leqslant \frac{d}{dt}\big(\big\|v(t)\big\|^2+\big\|w(t)\big\|^2\big)\\
&= 2\left(-\left(\nu+\frac{\kappa}{2}\right)\|\nabla v\|^2 - \gamma\|\nabla w\|^2 - 2\kappa\|w\|^2 + 2\kappa\int_{R^2} w\nabla\times v\,dx\right)\\
&\leqslant 2\left(-\left(\nu+\frac{\kappa}{2}\right)\|\nabla v\|^2 - \gamma\|\nabla w\|^2 - 2\kappa\|w\|^2 + 2\kappa\|w\|\|\nabla v\|\right)\\
&\leqslant -4\sigma_1\|\nabla u\|^2 \quad \text{(by Young inequality).}
\end{aligned} \tag{3.3}$$

Hence from (3.3), it follows that

$$\big\|u(t)\big\| \leqslant \big\|u(s)\big\| \quad \text{and} \tag{3.4}$$

$$\int_s^t \big\|\nabla u(\tau)\big\|^2\,d\tau \leqslant \frac{1}{4\sigma_1}\big\|u(s)\big\|^2, \quad \text{for } t\geqslant s\geqslant 0. \tag{3.5}$$

Similarly, by taking the L^2-inner product of (1.1) with $(\Delta v, \Delta w)$, one shows that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\big\|\nabla u(t)\big\|^2\\
&\quad\leqslant \frac{1}{2}\frac{d}{dt}\big(\big\|\nabla v(t)\big\|^2+\big\|\nabla w(t)\big\|^2\big)\\
&\quad= -\left(\nu+\frac{\kappa}{2}\right)\|\Delta v\|^2 - \gamma\|\Delta w\|^2 - 2\kappa\|\nabla w\|^2 - 2\kappa\int_{\mathbf{R}^2}(\Delta v)\cdot\nabla w\,dx\\
&\qquad+\int_{\mathbf{R}^2}(v\cdot\nabla v)\cdot\Delta v\,dx + \int_{\mathbf{R}^2}(v\cdot\nabla w)\cdot\Delta w\,dx
\end{aligned}$$

$$
\begin{aligned}
&\leqslant -\left(\nu+\frac{\kappa}{2}\right)\|\Delta v\|^2-\gamma\|\Delta w\|^2-2\kappa\|\nabla w\|^2+2\kappa\|\Delta v\|\|\nabla w\| \\
&\quad+\|u\|_\infty\|\nabla u\|\|\Delta u\| \quad \text{(by Hölder inequality)} \\
&\leqslant -\left(\nu+\frac{\kappa}{2}\right)\|\Delta v\|^2-\gamma\|\Delta w\|^2-2\kappa\|\nabla w\|^2+\frac{\kappa}{2}\|\Delta v\|^2+2\kappa\|\nabla w\|^2 \\
&\quad+\|u\|^{\frac{1}{2}}\|\nabla u\|\|\Delta u\|^{\frac{3}{2}} \quad \text{(by Gagliardo–Nirenberg inequality and Young inequality)} \\
&\leqslant -2\sigma_1\|\Delta u\|^2+c\|u_0\|^{\frac{1}{2}}\|\nabla u\|\|\Delta u\|^{\frac{3}{2}}\leqslant c\|\nabla u\|^4 \quad \big(\text{by (3.4) and Young inequality}\big).
\end{aligned}
$$

This gives

$$
\frac{\frac{d}{dt}\|\nabla u\|^2}{\|\nabla u\|^2}\leqslant c\|\nabla u\|^2. \tag{3.6}
$$

Integrating in time from s to t, for $t\geqslant s\geqslant 0$, we have

$$
\begin{aligned}
\big\|\nabla u(t)\big\|^2 &\leqslant \exp\left\{c\int_s^t \big\|\nabla u(\tau)\big\|^2\,d\tau\right\}\big\|\nabla u(s)\big\|^2 \\
&\leqslant \exp\left\{c\int_0^t \big\|\nabla u(\tau)\big\|^2\,d\tau\right\}\big\|\nabla u(s)\big\|^2 \\
&\leqslant e^{\frac{c}{4\sigma_1}\|u_0\|^2}\big\|\nabla u(s)\big\|^2=c_0\big\|\nabla u(s)\big\|^2,
\end{aligned} \tag{3.7}
$$

and hence

$$
\frac{t}{2}\big\|\nabla u(t)\big\|^2\leqslant c_0\int_{\frac{t}{2}}^t \|\nabla u(s)w\|^2\,ds\to 0 \quad \text{as } t\to\infty,
$$

which gives the validity of (3.1).

To prove (3.2), we now consider the integral equation formulation of (1.3)

$$
u(t)=e^{-At}u_0+\int_0^t e^{-A(t-s)}P(v\cdot\nabla u)\,ds. \tag{3.8}
$$

Observing that

$$
\varphi(t)\equiv t^{\frac{1}{2}}\big\|\nabla u(t)\big\|\to 0 \quad (t\to\infty) \tag{3.9}
$$

due to (3.1) and then applying (2.3), (2.5) and (3.9) to (3.8), we have

$$
\begin{aligned}
\big\|u(t)\big\| &= \left\|e^{-At}u_0+\int_0^t e^{-A(t-s)}P(v\cdot\nabla u)\,ds\right\| \\
&\leqslant \big\|e^{-At}u_0\big\|+\int_0^t \big\|Pe^{-A(t-s)}(v\cdot\nabla u)\big\|\,ds
\end{aligned}
$$

$$\leqslant c(1+t)^{-\frac{\gamma}{2}} + c\int_0^t (t-s)^{-\frac{1}{2}} \|v\cdot\nabla u\|_1\,ds$$

$$\leqslant c(1+t)^{-\frac{\gamma}{2}} + c\int_0^t (t-s)^{-\frac{1}{2}} \|\nabla u\|\|u\|\,ds$$

$$\leqslant c(1+t)^{-\frac{\gamma}{2}} + c\int_0^t (t-s)^{-\frac{1}{2}} (1+s)^{-\frac{1}{2}}\varphi(s)\big\|u(s)\big\|\,ds. \tag{3.10}$$

Since $0<\mu<1-\gamma/2$, one shows that

$$(1+t)^{\mu}\big\|u(t)\big\| \leqslant c + cJ(t)\sup_{0\leqslant s\leqslant t}(1+s)^{\mu}\big\|u(s)\big\|, \tag{3.11}$$

where

$$J(t) = (1+t)^{\mu}\int_0^t (t-s)^{-\frac{1}{2}}(1+s)^{-\frac{1}{2}-\mu}\varphi(s)\,ds.$$

Thus (3.2) holds true whenever $cJ(t)<1/2$, for large t, or

$$J(t)\to 0 \quad \text{as } t\to\infty.$$

Indeed, for a given small $\varepsilon>0$, we choose a constant $T>0$ so that $\varphi(t)<\varepsilon$ $(t>T)$. We thus have, for $t>T$,

$$\begin{aligned}
J(t) &= (1+t)^{\mu}\left\{\int_0^T + \int_T^t\right\}(t-s)^{-\frac{1}{2}}(1+s)^{-\frac{1}{2}-\mu}\varphi(s)\,ds \\
&\leqslant c(1+t)^{\mu}(t-T)^{-\frac{1}{2}}\int_0^T (1+s)^{-\frac{1}{2}-\mu}\,ds \\
&\quad + \varepsilon(1+t)^{\mu}\int_T^t (t-s)^{-\frac{1}{2}}(1+s)^{-\frac{1}{2}-\mu}\,ds \\
&\leqslant ct^{-(\frac{1}{2}-\mu)} + c\varepsilon \to c\varepsilon \quad \text{as } t\to\infty.
\end{aligned}$$

This completes the proof of Lemma 3.1. □

4. Derivation on the upper bounds

Considering the mild formulation of higher order derivative of (1.1)

$$\nabla^m u(t) = \nabla^m e^{-At}u_0 + \int_0^t \nabla^m e^{-A(t-s)}P(v\cdot\nabla u)\,ds, \tag{4.1}$$

and then applying (2.3)–(2.5), (3.9) to (4.1), one shows that

$$\begin{aligned}
\left\|\nabla^m u(t)\right\| &= \left\|\nabla^m e^{-At}u_0\right\| + \int_0^t \left\|\nabla^m e^{-A(t-s)}P(v\cdot\nabla u)\right\| ds \\
&= \left\|\nabla^m e^{-At}u_0\right\| + \int_0^t \left\|P\nabla^m e^{-A(t-s)}(v\cdot\nabla u)\right\| ds \\
&\leqslant \left\|\nabla^m e^{-At}u_0\right\| + \sum_{i=1}^2 \int_0^{\frac{t}{2}} \left\|\nabla^{m+1} e^{-A(t-s)}(v_i u)\right\| ds \\
&\quad + \sum_{i=1}^2 \int_{\frac{t}{2}}^t (t-s)^{-\frac{3}{4}} \left\|\nabla^m (v_i u)\right\|_{L^{\frac{4}{3}}} ds \\
&\leqslant c(1+t)^{-\frac{m+\gamma}{2}} + c\int_0^{\frac{t}{2}} (t-s)^{-\frac{m+2}{2}} \|u\|^2 ds \\
&\quad + c\int_{\frac{t}{2}}^t (t-s)^{-\frac{3}{4}} \|u\|^{\frac{1}{2}} \|\nabla u\|^{\frac{1}{2}} \left\|\nabla^m u\right\| ds \\
&\leqslant c(1+t)^{-\frac{m+\gamma}{2}} + c\int_0^{\frac{t}{2}} (t-s)^{-\frac{m+2}{2}} (1+s)^{-\mu} \|u\| ds \\
&\quad + c\int_{\frac{t}{2}}^t (t-s)^{-\frac{3}{4}} (1+s)^{-\frac{1+2\mu}{4}} \left\|\nabla^m u\right\| ds. \qquad (4.2)
\end{aligned}$$

Here we have used the inequality

$$\begin{aligned}
\left\|\nabla^m (v_i u)\right\|_{\frac{4}{3}} &\leqslant 2\|u\|_4 \left\|\nabla^m u\right\| + \sum_{k=1}^{m-1} \left\|\nabla^{m-k} u\right\|_4 \left\|\nabla^k u\right\| \\
&\leqslant c\|u\|^{\frac{1}{2}} \|\nabla u\|^{\frac{1}{2}} \left\|\nabla^m u\right\| \qquad (4.3)
\end{aligned}$$

due to the Hölder inequality and the Gagliardo–Nirenberg inequality (see, for example, [29]).

When $m = 0$, it follows from (4.2) that

$$(1+t)^{\frac{\gamma}{2}} \left\|u(t)\right\| \leqslant c + J_0(t) \sup_{0\leqslant s\leqslant t} (1+s)^{\frac{\gamma}{2}} \left\|u(s)\right\| \leqslant c + \frac{1}{2} \sup_{0\leqslant s\leqslant t} (1+s)^{\frac{\gamma}{2}} \left\|u(s)\right\|,$$

for t sufficiently large. Here the function

$$J_0(t) = c(1+t)^{\frac{\gamma}{2}} \int_0^{\frac{t}{2}} (t-s)^{-1} (1+s)^{-\mu-\frac{\gamma}{2}} ds$$

$$+ c(1+t)^{\frac{\gamma}{2}} \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{3}{4}} (1+s)^{-\frac{1}{4}-\gamma}\, ds \to 0 \tag{4.4}$$

as $t \to \infty$. Thus the estimate

$$\left\| u(t) \right\| \leqslant c(1+t)^{-\frac{\gamma}{2}} \tag{4.5}$$

holds true.

Similarly, the upper bound with $m > 0$ follows from the estimate

$$(1+t)^{\frac{m+\gamma}{2}} \left\| \nabla^m u(t) \right\| \leqslant c + J_m(t) \sup_{0 \leqslant s \leqslant t} (1+s)^{\frac{m+\gamma}{2}} \left\| \nabla^m u(s) \right\|$$

due to (4.2) and the property $J_m(t) \to 0$ as $t \to \infty$, where

$$\begin{aligned} J_m(t) &= c(1+t)^{\frac{m+\gamma}{2}} \int_0^{\frac{t}{2}} (t-s)^{-\frac{m+2}{2}} (1+s)^{-\mu-\frac{\gamma}{2}}\, ds \\ &\quad + c(1+t)^{\frac{m+\gamma}{2}} \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{3}{4}} (1+s)^{-\frac{1+2\mu}{4}-\frac{m+\gamma}{2}}\, ds \\ &\leqslant ct^{\frac{\gamma}{2}-1} \int_0^{\frac{t}{2}} (1+s)^{-\mu-\frac{\gamma}{2}}\, ds + c(1+t)^{-\frac{1+2\mu}{4}} \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{3}{4}}\, ds, \end{aligned} \tag{4.6}$$

for $t > 1$. Thus we have the upper bounds decay estimate as stated in Theorem 2.1,

$$\left\| \nabla^m u(t) \right\| \leqslant c(1+t)^{-\frac{m+\gamma}{2}}, \quad \text{for large } t \text{ and } m \geqslant 0. \tag{4.7}$$

5. Derivation on the lower bounds

Considering the error estimates of solutions between the micropolar fluid equations (1.3) and the linearized equations (1.4), we have from (4.1) that

$$\begin{aligned} &\left\| \nabla^m u(t) - \nabla^m e^{-At} u_0 \right\| \\ &\quad = \left\| \int_0^t \nabla^m e^{-A(t-s)} P(v \cdot \nabla u)\, ds \right\| \\ &\quad \leqslant \int_0^t \left\| P \nabla^m e^{-A(t-s)} (v \cdot \nabla u) \right\| ds \\ &\quad \leqslant \sum_{i=1}^{2} \int_0^{\frac{t}{2}} \left\| \nabla^{m+1} e^{-A(t-s)} (v_i u) \right\| ds + \sum_{i=1}^{2} \int_{\frac{t}{2}}^{t} (t-s)^{-\frac{3}{4}} \left\| \nabla^m (v_i u) \right\|_{L^{\frac{4}{3}}} ds \end{aligned}$$

$$\leqslant c\int_0^{\frac{t}{2}}(t-s)^{-\frac{m+2}{2}}\|u\|^2\,ds + c\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{3}{4}}\|u\|^{\frac{1}{2}}\|\nabla u\|^{\frac{1}{2}}\left\|\nabla^m u\right\|ds$$

$$\leqslant c\int_0^{\frac{t}{2}}(t-s)^{-\frac{m+2}{2}}(1+s)^{-\mu-\frac{\gamma}{2}}\,ds + c\int_{\frac{t}{2}}^{t}(t-s)^{-\frac{3}{4}}(1+s)^{-\frac{m+\gamma}{2}-\frac{1+2\gamma}{4}}\,ds$$

$$\leqslant c(1+t)^{-\frac{m+\gamma}{2}-\mu} + c(1+t)^{-\frac{m+\gamma}{2}-\frac{\gamma}{2}}, \quad \text{for large } t, \tag{5.1}$$

where the use is made of (2.3), (2.4), (3.2), (4.3), (4.7).

Hence, by (2.5), we have

$$\begin{aligned}\left\|\nabla^m u(t)\right\| &\geqslant \left\|\nabla^m e^{-At}u_0\right\| - \left\|\nabla^m u(t) - \nabla^m e^{-At}u_0\right\| \\ &\geqslant c(1+t)^{-\frac{m+\gamma}{2}}, \quad \text{for large } t,\end{aligned} \tag{5.2}$$

which proves the lower bounds on the decay of $\|\nabla^m u(t)\|$ as stated in Theorem 1.1. The proof of Theorem 1.1 is complete. □

Acknowledgment

The authors would like to express their gratitude to the referee for his/her valuable comments and suggestions.

References

[1] W. Borchers, T. Miyakawa, L^2 decay rate for the Navier–Stokes flow in half spaces, Math. Ann. 282 (1988) 139–155.

[2] W. Borchers, T. Miyakawa, Algebraic L^2 decay for the Navier–Stokes flows in exterior domains, Acta Math. 165 (1990) 189–227.

[3] A. Carpio, Large time behavior in incompressible Navier–Stokes equations, SIAM J. Math. Anal. 27 (1996) 449–475.

[4] Z.-M. Chen, A sharp decay result on strong solutions of the Navier–Stokes equations in the whole space, Comm. Partial Differential Equations 16 (1991) 801–820.

[5] J. Chen, Z.-M. Chen, B.-Q. Dong, Existence of H^2-global attractors of two-dimensional micropolar fluid flows, J. Math. Anal. Appl. 322 (2006) 512–522.

[6] B.-Q. Dong, Z.-M. Chen, Global attractors of two-dimensional micropolar fluid flows in some unbounded domains, Appl. Math. Comput. 182 (2006) 610–620.

[7] B.-Q. Dong, Y. Li, Large time behavior to the system of incompressible non-Newtonian fluids in $\mathbf{R}^2$, J. Math. Anal. Appl. 298 (2004) 667–676.

[8] B.-Q. Dong, Decay of solutions to equations modelling incompressible bipolar non-Newtonian fluids, Electron. J. Differential Equations 125 (2005) 1–13.

[9] B.-Q. Dong, Z.-M. Chen, Time decay rates of non-Newtonian flows in R^n_+, J. Math. Anal. Appl. 324 (2006) 820–833.

[10] A.C. Eringen, Theory of micropolar fluids, J. Math. Mech. 16 (1966) 1–18.

[11] G.P. Galdi, S. Rionero, A note on the existence and uniqueness of solutions of micropolar fluid equations, Internat. J. Engrg. Sci. 14 (1977) 105–108.

[12] T. Gallay, C. Wayne, Invariant manifolds and the long-time asymptotics of the Navier–Stokes and vorticity equations on $\mathbf{R}^2$, Arch. Ration. Mech. Anal. 163 (2002) 209–258.

[13] C. He, T. Miyakawa, Nonstationary Navier–Stokes flows in a two-dimensional exterior domain with rotational symmetries, Indiana Univ. Math. J. 55 (2006) 1483–1556.

[14] R. Kajikiya, T. Miyakawa, On L^2 decay of weak solutions of Navier–Stokes equations in R^n, Math. Z. 192 (1986) 135–148.

[15] T. Kato, Strong L^p solutions of the Navier–Stokes equations in R^m, with application to weak solutions, Math. Z. 187 (1984) 471–480.
[16] O.A. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible Fluids, Gordon & Breach, New York, 1969.
[17] H. Lange, The existence of instationary flows of incompressible micropolar fluids, Arch. Mech. 29 (1977) 741–744.
[18] J. Leray, Essai sur les le mouvement d'un liquide visqueux emplissant l'espace, Acta Math. 63 (1934) 193–248.
[19] G. Łukaszewicz, Micropolar Fluids. Theory and Applications, Modeling and Simulation in Science, Engineering and Technology, Birkhäuser, Boston, 1999.
[20] G. Łukaszewicz, Long time behavior of 2D micropolar fluid flows, Math. Comput. Model. 34 (2001) 1787–1802.
[21] G. Łukaszewicz, W. Sadowski, Uniform attractor for 2D magneto-micropolar fluid flow in some unbounded domains, Z. Angew. Math. Phys. 55 (2004) 247–257.
[22] T. Miyakawa, On upper and lower bounds of rates of decay for nonstationary Navier–Stokes flows in the whole space, Hiroshima Math. J. 32 (2002) 431–462.
[23] M. Oliver, E.S. Titi, Remark on the rate of decay of higher order derivatives for solutions to the Navier–Stokes equations in R^n, J. Funct. Anal. 172 (2000) 1–18.
[24] B. Resndiz, M. Rojas-Medar, Existence of weak solution of micropolar fluid equations in a time dependent domain, Rev. Mat. Apl. 23 (2002) 27–46.
[25] V.A. Sava, On the existence of solutions of the micropolar fluid equations, Rev. Roumaine Math. Pures Appl. 25 (1980) 1111–1123.
[26] M.E. Schonbek, Large time behavior of solutions to the Navier–Stokes equations, Comm. Partial Differential Equations 11 (1986) 733–763.
[27] M.E. Schonbek, Lower bounds of rates of decay for solution to Navier–Stokes equations, J. Amer. Math. Soc. 4 (1991) 423–449.
[28] M.E. Schonbek, Large time behaviour of solutions to the Navier–Stokes equations in H^m spaces, Comm. Partial Differential Equations 20 (1995) 103–117.
[29] M. Taylor, Pseudo Differential Operator and Nonlinear PDE, Birkhäuser Boston, Cambridge, 1991.
[30] R. Teman, The Navier–Stokes Equations, North-Holland, Amsterdam, 1977.
[31] M. Wiegner, Decay results for weak solutions of the Navier–Stokes equations in R^n, J. London Math. Soc. 35 (1987) 303–313.